ATD

2011

MATHEMATICS (Optional)

Paper – I

330041

Standard : Degree | **Total Marks : 200**

Nature : Conventional (Essay) type | **Duration : Three hours**

N.B. :

1) *Answers must be written in English.*
2) *Question **No. 1** is **compulsory**. Of the remaining questions, attempt **any Four** selecting one question from **each Section**.*
3) *Figures to the **RIGHT** indicate marks of the respective question.*
4) *Use of log table, Non-programmable calculator is permited, but any other Table / Code / Reference book are not permited.*
5) *Number of optional questions upto the prescribed number in the order in which they have been solved will only be assessed. Excess answers will not be assessed.*
6) *Credit will be given for orderly, concise and effective writing / presentation.*
7) *Candidates should not write roll number, any name (including their own), signature, address or any indication of their identity anywhere inside the answer book otherwise he/she will be penalised.*

1. Answer **any four** of the following : **Marks**

(a) If X_i and X_j are characteristic vectors corresponding to two distinct characteristic roots λ_i and λ_j respectively of a square matrix A of order n, show that X_i and X_j are always independent and are orthogonal if A is symmetric. **10**

(b) Show that $\int_0^{\pi/2} x^m \operatorname{cosec}^n x \, dx$ · **10**
(exists if and only if $n < (m + 1)$)

(c) Let $Y : \mathbb{R} \to \mathbb{R}^3$ be given by $Y(t) = (2\cos 3t, 2\sin 3t, 6t)$. Prove that Y(t) lies on a cylinder. Sketch Y(t), and $\dot{Y}(t)$. In which direction does $\dot{Y}(t)$ point ? **10**

(d) A wheel is rolling on a horizontal plane with its axis parallel to the plane. Give a set of generalised coordinates to describe this system. What are the constraints on this system ? **10**

(e) State the condition that a non-vertical line $y = mx + c$ is an asymptote to a curve $f(x,y) = 0$. Determine all the asymptotes of the cubic curve
$x^3 - x^2y + xy^2 - y^3 + 2x^2 + 8xy - 5y^2 + 2x + 7y = 0$. **10**

P.T.O.

Marks

SECTION – A

2. (a) Determine the dimension of the vector space $V = \{(x, y, z) \in \mathbb{R}^3 \mid x - y + z = 0, 2x + 5z = 0\}$. **10**

(b) Find non-singular matrices P and Q such that PAQ is in the normal form where

$$A = \begin{bmatrix} 1 & 2 & 3 & -2 \\ 2 & -2 & 1 & 3 \\ 3 & 0 & 4 & 1 \end{bmatrix}$$ and hence determine the rank of A. **10**

(c) Show that eigenvalues of a Hermitian matrix are real. **10**

(d) Define real quadratic form. Find rank, index and signature of the quadratic form $q(x, y, z) = x^2 + 2y^2 + 2z^2 - 2xy - 4yz + 2zx$. Hence determine whether $q(x, y, z)$ is positive definite-or negative definite or positive semi-definite or negative semi-definite. **10**

3. (a) Define sum of subspaces of a vector space. Show that if S_1 and S_2 are two sub-spaces of a vector space V, then $S_1 + S_2$ is the smallest sub-space of V containing S_1 and S_2. **10**

(b) Define congruent matrix. Show that the congruence relation on the set of all $n \times n$ matrices is an equivalence relation. **10**

(c) Investigate for what real values of λ and μ the simultaneous equations $x + y + 2z = 3$, $x + 2y + 3z = 5$, $x + 2y + \lambda z = \mu$ have

(i) no solution

(ii) unique solution

(iii) infinite number of solutions. **10**

(d) If A be a real skew -symmetric matrix of order n, show that $(I + A)$ is non-singular and $(I + A)^{-1} (I - A)$ is orthogonal. **10**

SECTION – B

4. (a) State and prove Rolle's theorem.

Let $f(x) = \begin{cases} 2 \text{ for } 0 \le x < 2 \\ 4 \text{ for } 2 \le x \le 4 \end{cases}$

Show that f(x) satisfies none of the conditions of Rolle's theorem, yet $f'(x) = 0$ for many points in [0, 4]. **15**

Marks

(b) Define stationary point of the function F(x, y). Determine absolute maximum and minimum of $F(x, y) = 4x + 3y$, given that $x^2 + y^2 \leq 4$. 15

(c) Find the area of the loop of the curve $x^3 + y^3 = 3axy$ where a is a constant. 10

5. (a) Show that limit of a real valued function is unique, if it exists. Prove that $f(x) = x \tan^{-1}\left(\frac{1}{x}\right)$ for $x \neq 0$ and $f(0) = 0$ is continuous but not differentiable at $x = 0$. 15

(b) If functions u, v, w of independent variables x, y, z are not independent, show that $\frac{\partial(u, v, w)}{\partial(x, y, z)} = 0$. Let $u = \frac{x+y}{1-xy}$ and $v = \tan^{-1}x + \tan^{-1}y$. Find $\frac{\partial(u, v)}{\partial(x, y)}$, if $xy \neq 1$. State whether u and v are functionally related. If so, find the relationship. 15

(c) Find the volume generated by revolution of the curve $y = \frac{a^3}{a^2 + x^2}$ about its asymptote, a is a constant. 10

SECTION - C

6. (a) Describe the spherical polar coordinates (r, θ, φ) in three dimensions. Draw the sets

(i) $r = 1$; (ii) $\theta = 1$ (iii) $\varphi = 1$. 10

(b) Find the equation of a plane in $\mathbb{R}^3$ passing through the points $a_1 = (1, 2, 3)$ $a_2 = (4, 5, 6)$ and $a_3 = (1, 1, 1)$. 15

(c) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be given by $f(x, y, z) = x^2 + 2y^2 + 3z^2$. 15

(i) Draw the level sets $f(x, y, z) = c$, for $c = 2, 3, 7$.

(ii) Find grad f (1, 2, 3).

(iii) Prove that grad f(1, 2, 3) is the direction in which f is increasing most rapidly.

7. (a) Let $f : \mathbb{R} \to \mathbb{R}$ be $f(x) = \exp(-x^2)$. 10

(i) Find the local extrema, regions of monotonicity and convexity.

(ii) Draw the graph of f.

(iii) Sketch the set $S = \{(x, y, z) \in \mathbb{R}^3 \mid z = \exp(-x^2 - y^2)\}$.

(b) Find an equation which represents a right circular cone with vertex angle $\pi/2$. 15

(c) Write down a vector field F in the plane such that $F(0, 0) = (0, 0)$ and F(a, b) is tangent to the circle $C = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = a^2 + b^2\}$ at (a, b) and which points in the anticlockwise direction. 15

Marks

SECTION – D

8. (a) Find the work done by the force $F(x, y, z) = (3y, 2x, 4z)$ over the path Y from $A = (0, 0, 0)$ to $B = (1, 1, 1)$ given by $Y(t) = (t, t^2, t^3)$, $0 \leq t \leq 1$. Is the work done from A to B independent of the path ? 15

(b) (i) When is a system said to be in equilibrium ?

(ii) Give an example of a system in a stable equilibrium.

(iii) Give an example of a system in equilibrium, which is not in a stable equilibrium. 15

(c) (i) What are Hamilton's equations of motion ?

(ii) Write down Hamiltons equations of motion for the simple pendulum with fixed length l and bob mass m. 10

9. (a) (i) Prove that the λ earth move around the sun in a plane.

(ii) Hence or otherwise, prove Kepler's second law. 15

(b) (i) State the principle of virtual work. Consider a simple pendulum with fixed length l.

(ii) What is a virtual displacement in this case ?

(iii) What are the forces acting on a simple pendulum ? Are they in equilibrium ? 15

(c) The Hamiltonian for a physical system is given by

$$H = \frac{1}{2}\sum_{1=1}^{3}(p_i^2 + q_i^2)$$

(i) Write down the equations of motion.

(ii) Prove that $F(q, p) = q_2 p_3 - q_3 p_2$ is a constant of the motion.

(iii) Are there any other constants of motion ? 10